

A law of the iterated logarithm sublinear expectations ^{*}

By Zengjing Chen, Feng Hu
Shandong University and Ajou University

Abstract In this paper, with the notion of independent identically distributed (IID) random variables under sub-linear expectations initiated by Peng, we investigate a law of the iterated logarithm for capacities. It turns out that our theorem is a natural extension of the Kolmogorov and the Hartman-Wintner laws of the iterated logarithm.

Keywords capacity · law of the iterated logarithm · IID · sub-linear expectation
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1 Introduction

The classical laws of the iterated logarithm (LIL) as fundamental limit theorems in probability theory play an important role in the development of probability theory and its applications. The original statement of the law of the iterated logarithm obtained by Khinchine (1924) is for a class of Bernoulli random variables. Kolmogorov (1929) and Hartman-Wintner (1941) extended Khinchine's result to large classes of independent random variables. Lévy (1937) extended Khinchine's result to martingales, an important class of dependent random variables; Strassen (1964) extended Hartman-Wintner's result to large classes of functional random variables. After that, the research activity of LIL has enjoyed both a rich classical period and a modern resurgence (see, Stout 1974 for details). To extend the LIL, a lot of fairly neat methods have been found (see, for example, De Acosta 1983), however, the key in the proofs of LIL is the additivity of the probabilities and the expectations. In practice, such additivity assumption is not feasible in many areas of applications because the uncertainty phenomena can not be modeled using additive probabilities or additive expectations. As an alternative to the traditional probability/expectation, capacities or nonlinear probabilities/expectations have been studied in many fields such as statistics, finance and economics. In statistics, capacities have been applied in robust statistics (Huber,1981). For example, under the assumption of 2-alternating capacity, Huber and Strassen (1973) have generalized the Neyman-Pearson lemma. Similarly Wasserman and Kadane (1990) have generalized the Bayes theorem for

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capacities. It is well-known that, in finance, an important question is how to calculate the price of contingent claims. The famous Black-Sholes's formula states that, if a market is complete and self-financial, then there exists a neutral probability measure P such that the pricing of any discounted contingent claim ξ in this market is given by $E_P[\xi]$. In this case, by Kolmogorov's strong law of large number and LIL, one can obtain the estimates of the mean $\mu := E_P[\xi]$ and the variance $\sigma^2 := E_P[|\xi - \mu|^2]$ with probability one by

$$\mu = \lim_{n \rightarrow \infty} \frac{1}{n} S_n, \quad \sigma = \limsup_{n \rightarrow \infty} (2n \log \log n)^{-1/2} |S_n - n\mu|$$

where S_n is the sum of the first n of a sample $\{X_i\}$ with mean μ and variance σ^2 . Statistically, an important feature of strong LLN and LIL is to provide a frequentist perspective for mean μ and standard variance σ . However, if the market is incomplete, such a neutral probability measure is no longer unique, it is a set \mathcal{P} of probability measures. In that case, one can give sub-hedge pricing and super-hedge pricing by $\mathcal{E}[\xi] := \inf_{Q \in \mathcal{P}} E_Q[\xi]$ and $\mathbb{E}[\xi] := \sup_{Q \in \mathcal{P}} E_Q[\xi]$. Obviously, both $\mathcal{E}[\cdot]$ and $\mathbb{E}[\cdot]$ as functional operators of random variables are nonlinear. Statistically, how to calculate sub-super hedge pricing is of interest. Motivated by sub-hedge and super-hedge pricing and model uncertainty in finance, Peng (2006-2009) initiated the notion of IID random variables and the definition of G -normal distribution. He further obtained new central limit theorems (CLT) under sub-linear expectations. Chen (2009) also obtained strong laws of large numbers in this framework. A natural question is the following: Can the classical LIL be generalized for capacities? In this paper, adapting the Peng's IID notion and applying Peng's CLT under sub-linear expectations, we investigate LIL for capacities. Our result shows that in the nonadditive setting, the supremum limit points of $\{(2n \log \log n)^{-1/2} |S_n|\}_{n \geq 3}$ lie, with probability (capacity) one, between the lower and upper standard variances, the others lie, with probability (capacity) one, between zero and the lower standard variance. This becomes the Kolmogorov and the Hartman-Wintner law of the iterated logarithm if capacity is additive, since in this case lower and upper variances coincide.

2 Notations and Lemmas

In this section, we introduce some basic notations and lemmas. For a given set \mathcal{P} of multiple prior probability measures on (Ω, \mathcal{F}) , let \mathcal{H} be the set of random variables on (Ω, \mathcal{F}) .

For any $\xi \in \mathcal{H}$, we define a pair of so-called maximum-minimum expectations $(\mathbb{E}, \mathcal{E})$ by

$$\mathbb{E}[\xi] := \sup_{P \in \mathcal{P}} E_P[\xi], \quad \mathcal{E}[\xi] := \inf_{P \in \mathcal{P}} E_P[\xi].$$

Without confusion, here and in the sequel, $E_P[\cdot]$ denotes the classical expectation under probability measure P . We use $\mathbb{E}[\cdot]$ to denote supremum expectation over \mathcal{P} .

Let $\xi = I_A$ for $A \in \mathcal{F}$, immediately, a pair (\mathbb{V}, v) of capacities is given by

$$\mathbb{V}(A) := \sup_{P \in \mathcal{P}} P(A), \quad v(A) := \inf_{P \in \mathcal{P}} P(A), \quad \forall A \in \mathcal{F}.$$

Obviously, \mathbb{E} is a sub-linear expectation in the sense that

Definition 1 A functional \mathbb{E} on $\mathcal{H} \mapsto (-\infty, +\infty)$ is called a sub-linear expectation, if it satisfies the following properties: for all $X, Y \in \mathcal{H}$,

- (a) Monotonicity: $X \geq Y$ implies $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
- (b) Constant preserving: $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$.
- (c) Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
- (d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.

Remark Artzner, Delbaen, Eber and Heath (1997) showed that a sub-linear expectation indeed is a supremum expectation. That is, if $\hat{\mathbb{E}}$ is a sub-linear expectation on \mathcal{H} ; then there exists a set (say $\hat{\mathcal{P}}$) of probability measures such that

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \hat{\mathcal{P}}} E_P[\xi], \quad -\hat{\mathbb{E}}[-\xi] = \inf_{P \in \hat{\mathcal{P}}} E_P[\xi].$$

Moreover, a sub-linear expectation $\hat{\mathbb{E}}$ can generate a pair $(\hat{\mathbb{V}}, \hat{v})$ of capacities denoted by

$$\hat{\mathbb{V}}(A) := \hat{\mathbb{E}}[I_A], \quad \hat{v}(A) := -\hat{\mathbb{E}}[-I_A], \quad \forall A \in \mathcal{F}.$$

Therefore, without confusion, we sometimes call the supremum expectation as the sub-linear expectation.

It is easy to check that the pair of capacities satisfies

$$\mathbb{V}(A) + v(A^c) = 1, \quad \forall A \in \mathcal{F}$$

where A^c is the complement set of A .

For ease of exposition, in this paper, we suppose that \mathbb{V} and v are continuous in the sense that

Definition 2 A set function $V: \mathcal{F} \rightarrow [0, 1]$ is called a continuous capacity if it satisfies

- (1) $V(\phi) = 0, V(\Omega) = 1$.
- (2) $V(A) \leq V(B)$, whenever $A \subset B$ and $A, B \in \mathcal{F}$.
- (3) $V(A_n) \uparrow V(A)$, if $A_n \uparrow A$.
- (4) $V(A_n) \downarrow V(A)$, if $A_n \downarrow A$, where $A_n, A \in \mathcal{F}$.

The following is the notion of IID random variables under sub-linear expectations introduced by Peng [12, 13, 14, 15, 16].

Definition 3 (IID under sublinear expectations/capacities)

Independence: Suppose that Y_1, Y_2, \dots, Y_n is a sequence of random variables such that $Y_i \in \mathcal{H}$. Random variable Y_n is said to be independent of $X := (Y_1, \dots, Y_{n-1})$ under \mathbb{E} , if for each measurable function φ on \mathbb{R}^n with $\varphi(X, Y_n) \in \mathcal{H}$ and $\varphi(x, Y_n) \in \mathcal{H}$ for each $x \in \mathbb{R}^{n-1}$, we have

$$\mathbb{E}[\varphi(X, Y_n)] = \mathbb{E}[\bar{\varphi}(X)],$$

where $\bar{\varphi}(x) := \mathbb{E}[\varphi(x, Y_n)]$ and $\bar{\varphi}(X) \in \mathcal{H}$.

Identical distribution: Random variables X and Y are said to be identically distributed, denoted by $X \stackrel{d}{=} Y$, if for each measurable function φ such that $\varphi(X), \varphi(Y) \in \mathcal{H}$,

$$\mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(Y)].$$

IID random variables: A sequence of random variables $\{X_i\}_{i=1}^\infty$ is said to be IID, if $X_i \stackrel{d}{=} X_1$ and X_{i+1} is independent of $Y := (X_1, \dots, X_i)$ for each $i \geq 1$.

Pairwise independence: Random variable X is said to be pairwise independent to Y under capacity V , if for all subsets D and $G \in \mathcal{B}(\mathbb{R})$,

$$V(X \in D, Y \in G) = V(X \in D)V(Y \in G).$$

The following lemma shows the relation between Peng's independence and pairwise independence.

Lemma 1 Suppose that $X, Y \in \mathcal{H}$ are two random variables. \mathbb{E} is a sub-linear expectation and (\mathbb{V}, v) is the pair of capacities generated by \mathbb{E} . If random variable X is independent of Y under \mathbb{E} , then X also is pairwise independent of Y under capacities \mathbb{V} and v .

Proof If we choose $\varphi(x, y) = I_D(x)I_G(y)$ for \mathbb{E} , by the definition of Peng's independence, it is easy to obtain

$$\mathbb{V}(X \in D, Y \in G) = \mathbb{V}(X \in D)\mathbb{V}(Y \in G).$$

Similarly, if we choose $\varphi(x, y) = -I_D(x)I_G(y)$ for \mathbb{E} , it is easy to obtain

$$v(X \in D, Y \in G) = v(X \in D)v(Y \in G).$$

The proof is complete.

Borel-Cantelli Lemma is still true for capacity under some assumptions.

Lemma 2 Let $\{A_n, n \geq 1\}$ be a sequence of events in \mathcal{F} and (\mathbb{V}, v) be a pair of capacities generated by sub-linear expectation \mathbb{E} .

(1) If $\sum_{n=1}^{\infty} \mathbb{V}(A_n) < \infty$, then $\mathbb{V}\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\right) = 0$.

(2) Suppose that $\{A_n, n \geq 1\}$ are pairwise independent with respect to \mathbb{V} , i.e.,

$$\mathbb{V}\left(\bigcap_{i=1}^{\infty} A_i^c\right) = \prod_{i=1}^{\infty} \mathbb{V}(A_i^c).$$

If $\sum_{n=1}^{\infty} v(A_n) = \infty$, then

$$v\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\right) = 1.$$

Proof

$$\begin{aligned} 0 &\leq \mathbb{V}\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\right) \\ &\leq \mathbb{V}\left(\bigcup_{i=n}^{\infty} A_i\right) \\ &\leq \sum_{i=n}^{\infty} \mathbb{V}(A_i) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

The proof of (1) is complete.

$$\begin{aligned} 0 &\leq 1 - v\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\right) \\ &= 1 - \lim_{n \rightarrow \infty} v\left(\bigcup_{i=n}^{\infty} A_i\right) \\ &= \lim_{n \rightarrow \infty} [1 - v\left(\bigcup_{i=n}^{\infty} A_i\right)] \\ &= \lim_{n \rightarrow \infty} \mathbb{V}\left(\bigcap_{i=n}^{\infty} A_i^c\right) \\ &= \lim_{n \rightarrow \infty} \prod_{i=n}^{\infty} \mathbb{V}(A_i^c) \\ &= \lim_{n \rightarrow \infty} \prod_{i=n}^{\infty} (1 - v(A_i)) \\ &\leq \lim_{n \rightarrow \infty} \prod_{i=n}^{\infty} \exp(-v(A_i)) \\ &= \lim_{n \rightarrow \infty} \exp\left(-\sum_{i=n}^{\infty} v(A_i)\right) = 0. \end{aligned}$$

We complete the proof of (2).

Definition 4 (*G-normal distribution*, see Definition 10 in Peng [13]) Given a sub-linear expectation \mathbb{E} , a random variable $\xi \in \mathcal{H}$ with

$$\bar{\sigma}^2 = \mathbb{E}[\xi^2], \quad \underline{\sigma}^2 = \mathcal{E}[\xi^2]$$

is called a G -normal distribution, denoted by $\mathcal{N}(0; [\underline{\sigma}^2, \bar{\sigma}^2])$, if for any bounded Lipschitz function ϕ , writting $u(t, x) := \mathbb{E}[\phi(x + \sqrt{t}\xi)]$, $(t, x) \in [0, \infty) \times \mathbb{R}$, then u is the viscosity solution of PDE:

$$\partial_t u - G(\partial_{xx}^2 u) = 0, \quad u(0, x) = \phi(x),$$

where $G(x) := \frac{1}{2}(\bar{\sigma}^2 x^+ - \underline{\sigma}^2 x^-)$ and $x^+ := \max\{x, 0\}$, $x^- := (-x)^+$.

The following lemma can be found in Denis, Hu and Peng [4].

Lemma 3 Suppose that ξ is G -normal distributed by $\mathcal{N}(0; [\underline{\sigma}^2, \bar{\sigma}^2])$. Let P be a probability measure and ϕ be a bounded continuous function. If $\{B_t\}_{t \geq 0}$ is a P -Brownian motion, then

$$\mathbb{E}[\phi(\xi)] = \sup_{\theta \in \Theta} E_P \left[\phi \left(\int_0^1 \theta_s dB_s \right) \right], \quad \mathcal{E}[\phi(\xi)] = \inf_{\theta \in \Theta} E_P \left[\phi \left(\int_0^1 \theta_s dB_s \right) \right],$$

where

$$\Theta := \{ \{ \theta_t \}_{t \geq 0} : \theta_t \text{ is } \mathcal{F}_t\text{-adapted process such that } \underline{\sigma} \leq \theta_t \leq \bar{\sigma} \},$$

$$\mathcal{F}_t := \sigma\{B_s, 0 \leq s \leq t\} \vee \mathcal{N}, \quad \mathcal{N} \text{ is the collection of } P\text{-null subsets.}$$

For the sake of completeness, the sketched proof of Lemma 3 is given in Appendix A.

With the notion of IID under sub-linear expectations, Peng shows the central limit theorem under sub-linear expectations (see Theorem 5.1 in Peng [15]).

Lemma 4 (Central limit theorem under sub-linear expectations) Let $\{X_i\}_{i=1}^\infty$ be a sequence of IID random variables. We further assume that

$$\mathbb{E}[X_1] = \mathcal{E}[X_1] = 0.$$

Then the sequence $\{\bar{S}_n\}_{n=1}^\infty$ defined by

$$\bar{S}_n := \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i$$

converges in law to ξ , i.e.,

$$\lim_{n \rightarrow \infty} \mathbb{E}[\varphi(\bar{S}_n)] = \mathbb{E}[\varphi(\xi)],$$

for any continuous function φ satisfying the linear growth condition, where ξ is a G -normal distribution.

Remark 1 Suppose that $\mathbb{E}[X_1^2] = \bar{\sigma}^2$, $\bar{\sigma} > 0$ and φ is a convex function, then, we have,

$$\mathbb{E}[\varphi(\xi)] = \frac{1}{\sqrt{2\pi\bar{\sigma}^2}} \int_{-\infty}^{\infty} \varphi(y) \exp\left(-\frac{y^2}{2\bar{\sigma}^2}\right) dy.$$

3 Main results

In this section, we will prove the following LIL for capacities:

Theorem 1 Let $\{X_n\}_{n=1}^\infty$ be a sequence of bounded IID random variables for sub-linear expectation \mathbb{E} with zero means and bounded variances, i.e.,

$$(A.1) \quad \mathbb{E}[X_1] = \mathcal{E}[X_1] = 0,$$

$$(A.2) \quad \mathbb{E}[X_1^2] = \bar{\sigma}^2, \mathcal{E}[X_1^2] = \underline{\sigma}^2, \text{ where } 0 < \underline{\sigma} \leq \bar{\sigma} < \infty.$$

Denote $S_n = \sum_{i=1}^n X_i$. Then

(I)

$$v \left(\underline{\sigma} \leq \limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \leq \bar{\sigma} \right) = 1.$$

(II)

$$v \left(-\bar{\sigma} \leq \liminf_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \leq -\underline{\sigma} \right) = 1.$$

(III) Suppose that $C(\{x_n\})$ is the cluster set of a sequence of $\{x_n\}$ in \mathbb{R} , then

$$v \left(C \left(\left\{ \frac{S_n}{\sqrt{2n \log \log n}} \right\} \right) \setminus \left\{ \limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}}, \liminf_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \right\} = (-\underline{\sigma}, \underline{\sigma}) \right) = 1.$$

In order to prove Theorem 1, we need the following lemmas.

Lemma 5 Suppose ξ is distributed to G normal $\mathcal{N}(0; [\underline{\sigma}^2, \bar{\sigma}^2])$, where $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$. Let ϕ be a bounded continuous function. Furthermore, if ϕ is a positively even function, then, for any $b \in \mathbb{R}$,

$$e^{-\frac{b^2}{2\underline{\sigma}^2}} \mathcal{E}[\phi(\xi)] \leq \mathcal{E}[\phi(\xi - b)].$$

Proof Let P be a probability measure, $\{B_t\}_{t \geq 0}$ be a P -Brownian motion. Since ξ is distributed to G -normal, by Lemma 3, we have

$$\mathcal{E}[\phi(\xi)] = \inf_{\theta \in \Theta} E_P \left[\phi \left(\int_0^1 \theta_s dB_s \right) \right].$$

For any $\theta \in \Theta$, write $\tilde{B}_t := B_t - \int_0^t \frac{b}{\theta_s} ds$. By Girsanov's theorem, $\{\tilde{B}_t\}_{t \geq 0}$ is a Q -Brownian motion under Q denoted by

$$\frac{dQ}{dP} := e^{-\frac{1}{2} \int_0^1 (\frac{b}{\theta_s})^2 ds + \int_0^1 \frac{b}{\theta_s} dB_s}.$$

That is

$$\frac{dP}{dQ} = e^{-\frac{1}{2} \int_0^1 (\frac{b}{\theta_s})^2 ds - \int_0^1 \frac{b}{\theta_s} d\tilde{B}_s}.$$

Thus

$$\begin{aligned} E_P \left[\phi \left(\int_0^1 \theta_s dB_s - b \right) \right] &= E_P \left[\phi \left(\int_0^1 \theta_t d(B_t - \int_0^t \frac{b}{\theta_s} ds) \right) \right] \\ &= E_P \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \right] \\ &= E_Q \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \cdot e^{-\frac{1}{2} \int_0^1 (\frac{b}{\theta_s})^2 ds - \int_0^1 \frac{b}{\theta_s} d\tilde{B}_s} \right] \\ &\geq e^{-\frac{1}{2} (\frac{b}{\varrho})^2} E_Q \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \cdot e^{-\int_0^1 \frac{b}{\theta_s} d\tilde{B}_s} \right]. \end{aligned} \quad (1)$$

We now prove that if ϕ is even, then

$$E_Q \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \cdot e^{-\int_0^1 \frac{b}{\theta_s} d\tilde{B}_s} \right] \geq E_Q \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \right].$$

In fact, let $\bar{B}_t := -\tilde{B}_t$, then $\{\bar{B}_t\}_{t \geq 0}$ is also a Q -Brownian motion. Note that the assumption that function ϕ is even, therefore

$$\begin{aligned} E_Q \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \cdot e^{-\int_0^1 \frac{b}{\theta_s} d\tilde{B}_s} \right] &= E_Q \left[\phi \left(\int_0^1 \theta_s d\bar{B}_s \right) \cdot e^{-\int_0^1 \frac{b}{\theta_s} d\bar{B}_s} \right] \\ &= E_Q \left[\phi \left(-\int_0^1 \theta_s d\tilde{B}_s \right) \cdot e^{\int_0^1 \frac{b}{\theta_s} d\tilde{B}_s} \right] \\ &= E_Q \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \cdot e^{\int_0^1 \frac{b}{\theta_s} d\tilde{B}_s} \right]. \end{aligned}$$

Since $\frac{e^{\int_0^1 \frac{b}{\theta_s} d\tilde{B}_s} + e^{-\int_0^1 \frac{b}{\theta_s} d\tilde{B}_s}}{2} \geq 1$, we have

$$\begin{aligned} E_Q \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \cdot e^{-\int_0^1 \frac{b}{\theta_s} d\tilde{B}_s} \right] &= \frac{1}{2} E_Q \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \cdot \left(e^{\int_0^1 \frac{b}{\theta_s} d\tilde{B}_s} + e^{-\int_0^1 \frac{b}{\theta_s} d\tilde{B}_s} \right) \right] \\ &\geq E_Q \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \right]. \end{aligned}$$

From (1), we have

$$\begin{aligned} \mathcal{E}[\phi(\xi - b)] &= \inf_{\theta \in \Theta} E_P \left[\phi \left(\int_0^1 \theta_s dB_s - b \right) \right] \geq e^{-\frac{1}{2}(b/\varrho)^2} \inf_{\theta \in \Theta} E_Q \left[\phi \left(\int_0^1 \theta_s d\tilde{B}_s \right) \right] \\ &= e^{-\frac{1}{2}(b/\varrho)^2} \inf_{\theta \in \Theta} E_P \left[\phi \left(\int_0^1 \theta_s dB_s \right) \right] \\ &= e^{-\frac{1}{2}(b/\varrho)^2} \mathcal{E}[\phi(\xi)]. \end{aligned}$$

The proof of Lemma 5 is complete.

Lemma 6 Under the assumptions of Theorem 1, then, for each $r > 2$, there exists a positive constant K_r such that

$$\mathbb{E}[\max_{i \leq n} |S_{m,i}|^r] \leq K_r n^{\frac{r}{2}} \quad \text{for all } m \geq 0,$$

where $S_{m,n} = \sum_{i=m+1}^{m+n} X_i$.

Proof. First, we prove that there exists a positive constant C_r such that

$$\sup_{m \geq 0} \mathbb{E}|S_{m,n}|^r \leq C_r n^{\frac{r}{2}}. \quad (2)$$

By Lemma 4 and Remark 1, it is easy to check that

$$\lim_{n \rightarrow \infty} \mathbb{E}[|S_{m,n}/\sqrt{n}|^r] = \mathbb{E}[|\xi|^r] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} |y|^r \exp\left(-\frac{y^2}{2\sigma^2}\right) dy < \infty.$$

So, we can choose

$$D_r > \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} |y|^r \exp\left(-\frac{y^2}{2\sigma^2}\right) dy,$$

then there exists n_0 such that $\forall n \geq n_0$,

$$\mathbb{E}|S_{m,n}|^r \leq D_r n^{\frac{r}{2}}.$$

Note that $\{X_n\}_{n=1}^{\infty}$ is a bounded sequence, then there exists a constant $M > 0$, such that, for each n , $|X_n| \leq M$. So we can obtain (2) holds. Hence, in a manner similar to Theorem 3.7.5 of Stout [17], we can obtain

$$\mathbb{E}[\max_{i \leq n} |S_{m,i}|^r] \leq K_r n^{\frac{r}{2}} \quad \text{for all } m \geq 0.$$

Lemma 7 Under the assumptions of Theorem 1, if

$$v\left(\limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{2n \log \log n}} \leq \bar{\sigma}\right) = 1,$$

then, for any $b \in \mathbb{R}$ satisfying $|b| < \underline{\sigma}$,

$$v\left(\liminf_{n \rightarrow \infty} \left| \frac{S_n}{\sqrt{2n \log \log n}} - b \right| = 0\right) = 1.$$

Proof We only need to prove that for any $\epsilon > 0$,

$$v\left(\liminf_{n \rightarrow \infty} \left| \frac{S_n}{\sqrt{2n \log \log n}} - b \right| \leq \epsilon\right) = 1.$$

To do so, we only need to prove that there exists an increasing subsequence $\{n_k\}$ of $\{n\}$ such that

$$v\left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{|S_{n_k}/\sqrt{2n_k \log \log n_k} - b| \leq \epsilon\}\right) = 1. \quad (3)$$

Indeed, let us choose $n_k := k^k$ for $k \geq 1$. For each $t > 0$, write

$$N_k := [(n_{k+1} - n_k)^2 t^2 / 2n_{k+1} \log \log n_{k+1}],$$

$$m_k := [2n_{k+1} \log \log n_{k+1} / t^2 (n_{k+1} - n_k)],$$

$$r_k := \sqrt{2n_{k+1} \log \log n_{k+1}} / t m_k.$$

Since $\{X_n\}_{n=1}^\infty$ is a sequence of IID random variables under sub-linear expectation \mathbb{E} , we have

$$\begin{aligned} & v \left(\left| \frac{S_{n_{k+1}} - S_{n_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} - b \right| \leq \epsilon \right) = v \left(b - \epsilon \leq \frac{S_{n_{k+1}} - S_{n_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} \leq b + \epsilon \right) \\ & \geq v \left(b - \epsilon/2 \leq \frac{S_{N_k m_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} \leq b + \epsilon/2 \right) \cdot v \left(-\epsilon/2 \leq \frac{S_{n_{k+1} - n_k} - S_{N_k m_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} \leq \epsilon/2 \right) \\ & \geq v \left(t m_k (b - \epsilon/2) \leq \frac{S_{N_k m_k}}{r_k} \leq t m_k (b + \epsilon/2) \right) \cdot v \left(-\epsilon/2 \leq \frac{S_{n_{k+1} - n_k - N_k m_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} \leq \epsilon/2 \right) \\ & \geq \left(v \left(b t - \epsilon t/2 \leq \frac{S_{N_k}}{r_k} \leq b t + \epsilon t/2 \right) \right)^{m_k} \cdot v \left(-\epsilon/2 \leq \frac{S_{n_{k+1} - n_k - N_k m_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} \leq \epsilon/2 \right) \\ & \geq (\mathcal{E}[\phi(S_{N_k}/r_k - b t)])^{m_k} \cdot v \left(-\epsilon/2 \leq \frac{S_{n_{k+1} - n_k - N_k m_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} \leq \epsilon/2 \right), \end{aligned} \tag{4}$$

where $\phi(x)$ is a even function defined by

$$\phi(x) := \begin{cases} 1 - e^{|x| - \epsilon t/2}, & |x| \leq \epsilon t/2; \\ 0, & |x| > \epsilon t/2. \end{cases}$$

Note the fact that $N_k \rightarrow \infty$, as $k \rightarrow \infty$ and applying Lemmas 4 and 5,

$$\log \mathcal{E}[\phi(S_{N_k}/r_k - b t)] \rightarrow \log \mathcal{E}[\phi(\xi - b t)] \geq -\frac{b^2 t^2}{2\sigma^2} + \log \mathcal{E}[\phi(\xi)], \quad \text{as } k \rightarrow \infty.$$

Thus

$$\begin{aligned} & \log (\mathcal{E}[\phi(S_{N_k}/r_k - b t)])^{m_k} \cdot (n_{k+1} - n_k) / 2n_{k+1} \log \log n_{k+1} \\ & = (n_{k+1} - n_k) / 2n_{k+1} \log \log n_{k+1} \cdot m_k \log \mathcal{E}[\phi(S_{N_k}/r_k - b t)] \\ & \rightarrow t^{-2} \log \mathcal{E}[\phi(\xi - b t)] \geq -\frac{1}{2}(b/\sigma)^2 + t^{-2} \log \mathcal{E}[\phi(\xi)]. \end{aligned} \tag{5}$$

However,

$$\liminf_{t \rightarrow \infty} t^{-2} \log \mathcal{E}[\phi(\xi - b t)] \geq -\frac{1}{2}(b/\sigma)^2. \tag{6}$$

So, from (5) and (6), we have, for any $\delta > 0$ and large enough t ,

$$\lim_{k \rightarrow \infty} \log (\mathcal{E}[\phi(S_{N_k}/r_k - b t)])^{m_k} \cdot (n_{k+1} - n_k) / 2n_{k+1} \log \log n_{k+1} \geq -\frac{1}{2}(|b|/\sigma + \delta/2)^2. \tag{7}$$

On the other hand, by Chebyshev's inequality,

$$\mathbb{V} \left(\left| \frac{S_{n_{k+1} - n_k - N_k m_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} \right| > \epsilon/2 \right) \leq 2(n_{k+1} - n_k - N_k m_k) \bar{\sigma}^2 / \epsilon^2 n_{k+1} \log \log n_{k+1} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

So, as $k \rightarrow \infty$,

$$\begin{aligned} & (n_{k+1} - n_k) / 2n_{k+1} \log \log n_{k+1} \cdot \log v \left(-\epsilon/2 \leq \frac{S_{n_{k+1}} - S_{n_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} \leq \epsilon/2 \right) \\ &= (n_{k+1} - n_k) / 2n_{k+1} \log \log n_{k+1} \cdot \log \left(1 - \mathbb{V} \left(\left| \frac{S_{n_{k+1}} - S_{n_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} \right| > \epsilon/2 \right) \right) \rightarrow 0. \end{aligned} \quad (8)$$

Therefore, from (4), (7) and (8), we have

$$\liminf_{k \rightarrow \infty} (n_{k+1} - n_k) / 2n_{k+1} \log \log n_{k+1} \cdot \log v \left(\left| \frac{S_{n_{k+1}} - S_{n_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} - b \right| \leq \epsilon \right) \geq -\frac{1}{2}(|b|/\underline{\sigma} + \delta/2)^2.$$

Now we choose $\delta > 0$ such that $|b|/\underline{\sigma} + \delta < 1$. Then, for given $\delta > 0$, there exists k_0 such that $\forall k \geq k_0$,

$$\begin{aligned} & v \left(\left| \frac{S_{n_{k+1}} - S_{n_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} - b \right| \leq \epsilon \right) \\ & \geq \exp \{ -2n_{k+1} \log \log n_{k+1} / (n_{k+1} - n_k) \cdot ((|b|/\underline{\sigma} + \delta)/2) \} \\ & \sim \exp(-(|b|/\underline{\sigma} + \delta) \log \log n_{k+1}). \end{aligned}$$

Thus

$$\sum_{k=1}^{\infty} v \left(\left| \frac{S_{n_{k+1}} - S_{n_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} - b \right| \leq \epsilon \right) = \infty.$$

Using the second Borel-Cantelli Lemma, we have

$$\liminf_{k \rightarrow \infty} \left| \frac{S_{n_k} - S_{n_{k-1}}}{\sqrt{2n_k \log \log n_k}} - b \right| \leq \epsilon, \quad \text{a.s. } v.$$

But

$$\left| \frac{S_{n_k}}{\sqrt{2n_k \log \log n_k}} - b \right| \leq \left| \frac{S_{n_k} - S_{n_{k-1}}}{\sqrt{2n_k \log \log n_k}} - b \right| + \frac{|S_{n_{k-1}}|}{\sqrt{2n_{k-1} \log \log n_{k-1}}} \frac{\sqrt{2n_{k-1} \log \log n_{k-1}}}{\sqrt{2n_k \log \log n_k}}. \quad (9)$$

Note the following fact

$$\frac{n_{k-1}}{n_k} \rightarrow 0, \quad \text{as } k \rightarrow \infty$$

and

$$\limsup_{n \rightarrow \infty} |S_n| / \sqrt{2n \log \log n} \leq \bar{\sigma}, \quad \text{a.s. } v.$$

Hence, from inequality (9), for any $\epsilon > 0$,

$$\liminf_{k \rightarrow \infty} \left| \frac{S_{n_k}}{\sqrt{2n_k \log \log n_k}} - b \right| \leq \epsilon, \quad \text{a.s. } v,$$

therefore,

$$v \left(\liminf_{n \rightarrow \infty} \left| \frac{S_n}{\sqrt{2n \log \log n}} - b \right| \leq \epsilon \right) = 1.$$

Since ϵ is arbitrary, we have

$$v \left(\liminf_{n \rightarrow \infty} \left| \frac{S_n}{\sqrt{2n \log \log n}} - b \right| = 0 \right) = 1.$$

We complete the proof of Lemma 7.

The Proof of Theorem 1 (I) First, we prove that

$$v \left(\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \leq \bar{\sigma} \right) = 1.$$

For each $\epsilon > 0$ and $\lambda > 0$, by Markov's inequality,

$$\begin{aligned} \mathbb{V} \left(\frac{S_n}{\sqrt{2n \log \log n}} > (1 + \epsilon) \bar{\sigma} \right) &= \mathbb{V} \left(\frac{S_n}{\sqrt{n \bar{\sigma}^2}} > (1 + \epsilon) \sqrt{2 \log \log n} \right) \\ &\leq \exp(-2(1 + \epsilon)^2 \lambda \log \log n) \mathbb{E} \left[\exp \left(\lambda \left(\frac{S_n}{\sqrt{n \bar{\sigma}^2}} \right)^2 \right) \right]. \end{aligned} \quad (10)$$

On the other hand, by Lemma 4 and Remark 1, we have, if $\lambda < \frac{1}{2}$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\exp \left(\lambda \left(\frac{S_n}{\sqrt{n \bar{\sigma}^2}} \right)^2 \right) \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(\lambda y^2) \exp \left(-\frac{y^2}{2} \right) dy < \infty.$$

Fixing $\beta > 1$, for each $\epsilon > 0$, we can choose $\lambda_\epsilon \in (0, \frac{1}{2})$ such that $\beta = 2(1 + \epsilon)^2 \lambda_\epsilon > 1$. So, we can choose

$$C_\epsilon > \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(\lambda_\epsilon y^2) \exp \left(-\frac{y^2}{2} \right) dy,$$

then there exists n_0 such that $\forall n \geq n_0$,

$$E \left[\exp \left(\lambda_\epsilon \left(\frac{S_n}{\sqrt{n \bar{\sigma}^2}} \right)^2 \right) \right] \leq C_\epsilon. \quad (11)$$

From (10) and (11), we can obtain, $\forall n \geq n_0$,

$$\mathbb{V} \left(\frac{S_n}{\sqrt{2n \log \log n}} > (1 + \epsilon) \bar{\sigma} \right) \leq C_\epsilon \exp(-\beta \log \log n).$$

Choose $0 < \alpha < 1$ such that $\alpha\beta > 1$. Let $n_k := [e^{k^\alpha}]$ for $k \geq 1$. Then

$$\sum_{n_k \geq n_0} \mathbb{V} \left(\frac{S_{n_k}}{\sqrt{2n_k \log \log n_k}} > (1 + \epsilon) \bar{\sigma} \right) \leq D_\epsilon \sum_{n_k \geq n_0} k^{-\alpha\beta} < \infty,$$

where D_ϵ is a positive constant. By the first Borel-Cantelli Lemma, we can get

$$\mathbb{V} \left(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \left\{ \frac{S_{n_k}}{\sqrt{2n_k \log \log n_k}} > (1 + \epsilon) \bar{\sigma} \right\} \right) = 0.$$

Also

$$v \left(\limsup_{k \rightarrow \infty} \frac{S_{n_k}}{\sqrt{2n_k \log \log n_k}} \leq (1 + \epsilon) \bar{\sigma} \right) = 1.$$

Let $M_k := \max_{n_k \leq n < n_{k+1}} \frac{|S_n - S_{n_k}|}{\sqrt{2n_k \log \log n_k}}$ for $k \geq 1$. For each $k \geq 1$,

$$\frac{S_n}{\sqrt{2n \log \log n}} \leq \frac{S_{n_k}}{\sqrt{2n_k \log \log n_k}} \frac{\sqrt{2n_k \log \log n_k}}{\sqrt{2n \log \log n}} + M_k \frac{\sqrt{2n_k \log \log n_k}}{\sqrt{2n \log \log n}},$$

for $n_k \leq n < n_{k+1}$. For given α , we choose $p > 2$ such that $p(1 - \alpha) \geq 2$. By Lemma 6, we get

$$\sum_{k=1}^{\infty} \mathbb{E}[M_k^p] \leq K_p \sum_{k=1}^{\infty} \frac{(n_{k+1} - n_k)^{\frac{p}{2}}}{(2n_k \log \log n_k)^{\frac{p}{2}}} \leq D'_p \sum_{k=1}^{\infty} k^{-\frac{p(1-\alpha)}{2}} (\log k)^{-\frac{p}{2}} < \infty, \quad (12)$$

where D'_p is a positive constant. From (12) and by Chebyshev's inequality, for each $\epsilon > 0$,

$$\sum_{k=1}^{\infty} \mathbb{V}(M_k > \epsilon) \leq \sum_{k=1}^{\infty} \frac{\mathbb{E}[M_k^p]}{\epsilon^p} < \infty.$$

Hence, by the first Borel-Cantelli Lemma again,

$$v \left(\limsup_{k \rightarrow \infty} M_k \leq \epsilon \right) = 1.$$

Since ϵ is arbitrary, we have

$$v \left(\lim_{k \rightarrow \infty} M_k = 0 \right) = 1.$$

Noting that

$$\frac{\sqrt{2n_k \log \log n_k}}{\sqrt{2n_{k+1} \log \log n_{k+1}}} \rightarrow 1, \quad \text{as } k \rightarrow \infty,$$

we have

$$\begin{aligned} & v \left(\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \leq (1 + \epsilon) \bar{\sigma} \right) \\ & \geq v \left(\left\{ \limsup_{k \rightarrow \infty} \frac{S_{n_k}}{\sqrt{2n_k \log \log n_k}} \leq (1 + \epsilon) \bar{\sigma} \right\} \cap \left\{ \lim_{k \rightarrow \infty} M_k = 0 \right\} \right) \\ & = 1, \end{aligned}$$

which implies

$$v \left(\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \leq \bar{\sigma} \right) = 1. \quad (13)$$

Similarly, considering the sequence $\{-X_n\}_{n=1}^{\infty}$, it suffices to obtain

$$v \left(\limsup_{n \rightarrow \infty} \frac{-S_n}{\sqrt{2n \log \log n}} \leq \bar{\sigma} \right) = 1.$$

Also

$$v \left(\liminf_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \geq -\bar{\sigma} \right) = 1. \quad (14)$$

Now we prove that

$$v \left(\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \geq \underline{\sigma} \right) = 1.$$

Indeed, from (13) and (14), it is easy to obtain

$$v \left(\limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{2n \log \log n}} \leq \bar{\sigma} \right) = 1.$$

For any number $b \in (0, \underline{\sigma})$, noting the fact that $|b| < \underline{\sigma}$, by Lemma 7, we have

$$v \left(\liminf_{n \rightarrow \infty} \left| \frac{S_n}{\sqrt{2n \log \log n}} - b \right| = 0 \right) = 1,$$

which implies that

$$v \left(\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \geq \underline{\sigma} \right) = 1. \quad (15)$$

So, from (13) and (15), we can obtain

$$v \left(\underline{\sigma} \leq \limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \leq \bar{\sigma} \right) = 1.$$

The proof of (I) is complete.

(II) Considering the sequence $\{-X_n\}_{n=1}^{\infty}$, by (I), it suffices to obtain

$$v \left(\underline{\sigma} \leq \limsup_{n \rightarrow \infty} \frac{-S_n}{\sqrt{2n \log \log n}} \leq \bar{\sigma} \right) = 1.$$

Thus

$$v \left(-\bar{\sigma} \leq \liminf_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} \leq -\underline{\sigma} \right) = 1.$$

To prove (III). We only need to prove that for any number $b \in (-\underline{\sigma}, \underline{\sigma})$,

$$v \left(\liminf_{n \rightarrow \infty} \left| \frac{S_n}{\sqrt{2n \log \log n}} - b \right| = 0 \right) = 1. \quad (16)$$

Noting the fact that $|b| < \underline{\sigma}$, by Lemma 7, we can easily obtain (16).

The proof of (III) is complete.

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School of Mathematics

Shandong University

Jinan, 250100, PRC.

Department of Financial Engineering

Ajou University

Suwon, KR.